Abstract

"Every continuous map $f \quad S^n \to \mathbb{R}^n$, $n \in \mathbb{N}$, takes identical values on a pair of antipodal points" This is the well known Borsuk-Ulam theorem The main aim of this project is to provide an algebraic proof this theorem by using projective algebraic sets In fact, we prove a more general form of Borsuk-Ulam theorem called the Borsuk's Nullstellensatz (see (4 2)) by establishing its equivalence with the real algebraic Nullstellensatz (see (4 4) and (10 3))

The proof of Borsuk-Ulam theorem for the case $n = 1$ is an easy application of the intermediate value theorem The case $n = 2$ is already non-trivial and its needs the concept of the first fundamental group which was introduced by Poincaré The general case is usually proved by using higher homology groups

Our proof of the real algebraic Nullstellensatz is elementary and uses standard definitions and basic properties of Poincaré series, projective dimension and multiplicity of (standard) graded algebras over a field From these considerations our proof of the real algebraic Nullstellensatz only uses the fundamental property of the real numbers, namely every odd degree polynomial over the field of real numbers has a real root We say a field K is a 2-field if every odd degree polynomial over K has a root in K Therefore the real algebraic Nullstellensatz can be generalized to an arbitrary 2-field K More precisely, we prove the following

(10 2)**Theorem**(Projective Nullstellensatz over a 2-field) Let A be a standard graded algebra over a 2-field K of projective dimension $d \geq 0$ with odd multiplicity e and $f_1, \quad, f_r \in A$ be homogeneous elements of odd degree with $r \leq d$ Then $f_1, \quad, f_r$ have a common zero in the projective space $\mathbb{P}_A(K)$ of A over K, i e

$$\emptyset \neq \mathbb{P}_{A/\mathfrak{A}}(K) = V_+(f_1, \quad, f_r) \subseteq \mathbb{P}_A(K), \; \mathfrak{A} = \sum_{i=1}^{i=r} Af_i$$

Further if A is an integral domain and $f \in A$ is a homogeneous element such that $f(\tau) = 0$ for every $\tau \in \mathbb{P}_A(K)$ then $f = 0$